

TED (10)-1015

Reg. No.

(REVISION-2010)

Signature

SECOND SEMESTER DIPLOMA EXAMINATION IN ENGINEERING/
TECHNOLOGY- MARCH, 2012

TECHNICAL MATHEMATICS- II

(Common – Except DCP and CABM)

[Time: 3 hours

(Maximum marks: 100)

Marks

PART –A

(Maximum marks: 10)

(Answer all questions. Each question carries 2 marks)

I.

(a) Evaluate $\lim_{x \rightarrow \infty} \frac{7-x}{3x+1}$

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{7-x}{3x+1} &= \lim_{x \rightarrow \infty} \frac{x\left(\frac{7}{x}-1\right)}{x\left(3+\frac{1}{x}\right)} \\ &= \frac{\left(\frac{7}{\infty}-1\right)}{\left(3+\frac{1}{\infty}\right)} = \frac{0-1}{3+0} \\ &= \frac{-1}{3}\end{aligned}$$

(b) Find $\frac{dy}{dx}$, if $y = \frac{1}{\sec\sqrt{x}}$

$$y = \frac{1}{\sec\sqrt{x}}$$

$$y = \cos\sqrt{x}$$

$$\frac{dy}{dx} = -\sin\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

(c) Find the slope if the curve $y = \frac{3}{x^2}$ at the point (1, 3)

$$\text{Slope} = \frac{dy}{dx}$$

$$= \frac{-6}{x^3}$$

$$\therefore \text{At (1, 3), slope} = \frac{-6}{1^3} = \frac{-6}{1} = -6$$

(d) Evaluate $\int_0^3 x(x^2 + 1)dx$

$$\begin{aligned}\int_0^3 x(x^2 + 1)dx &= \int_0^3 (x^3 + x)dx \\ &= \left[\frac{x^4}{4} + \frac{x^2}{2} \right]_0^3 \\ &= \frac{3^4}{4} + \frac{3^2}{2} - \frac{0^4}{4} - \frac{0^2}{2} \\ &= \frac{81}{4} + \frac{9}{2} - 0 \\ &= \frac{81+18}{4} \\ &= \frac{99}{4}\end{aligned}$$

(e) Solve $\frac{d^2y}{dx^2} = \sin x$

$$\frac{d^2y}{dx^2} = \sin x$$

Integrating both sides w.r.to x

$$\frac{dy}{dx} = -\cos x$$

Again integrating both sides w.r.to x

$$Y = -\sin x + c$$

PART -B

Answer any five questions. Each question carries 6 marks

II.

(a)

i. Evaluate $\lim_{\theta \rightarrow 0} \frac{\sin 3\theta \cos \theta}{\theta}$

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\sin 3\theta \cos \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\theta} \times \lim_{\theta \rightarrow 0} \cos \theta \\ &= 3 \times \cos \theta \\ &= 3 \times 1 \\ &= 3 \\ &= 1 \end{aligned}$$

ii. Find 'a' if $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$ is consistent at $x = 0$.

$$f(0) = a$$

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= 1 \end{aligned}$$

∴ Function is continuous at $x = 0$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

$$\text{ie } a = 1$$

(b) If $y = x^2 \sin x$, prove that $x^2 y'' - 4xy' + (x^2 + 6)y = 0$

$$y = x^2 \sin x$$

$$y' = x^2 \cos x + \sin x \cdot 2x$$

$$y'' = x^2 \cos x - \sin x + \cos x \cdot 2x + \sin x \cdot x \cdot 2 + 2x \cdot x \cdot \cos x$$

$$y'' = -x^2 \sin x + 2x \cos x + 2 \sin x + 2x \cos x$$

$$y'' = -x^2 \sin x + 4x \cos x + 2 \sin x$$

$$\therefore x^2 y'' - 4xy' + (x^2 + 6)y = \therefore x^2 y'' - 4xy' + (x^2 y + 6y) -$$

$$= x^2 (-x^2 \sin x + 4x \cos x + 2 \sin x)$$

$$- 4x(x^2 \cos x + \sin x \cdot 2x) + x^2 x^2 \sin x + 6x^2 \sin x$$

$$= -x^4 \sin x + 4x^3 \cos x + 2x^2 \sin x - 4x^3 \cos x - x^2 \sin x$$

$$+x^4 \sin x + 6x^2 \sin x$$

$$= 0$$

(c) A balloon is spherical in shape. Gas is escaping from it at the rate of 10 cc/sec. How fast is the surface area shrinking when the radius is 15cms?

$$V = \frac{4}{3} \pi r^3$$

$$\frac{dv}{dt} = \frac{4}{3} \pi 3r^2 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = 0$$

$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$R = 15\text{cm}$$

$$10 = 4\pi 15^2 \frac{dr}{dt}$$

$$10 = 4\pi 225 \frac{dr}{dt}$$

$$10 = 900\pi \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{10}{900\pi} = \frac{1}{90\pi}$$

$$\text{Surface area } S = 4\pi r^2$$

$$\frac{ds}{dt} = 4\pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\frac{ds}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

$$\frac{ds}{dt} = 8\pi r \cdot \frac{1}{90\pi}$$

$$\frac{ds}{dt} = 8 \times 15 \times \frac{1}{90}$$

$$= \frac{120}{90}$$

$$= \frac{4}{3}$$

\therefore Surface area shrinking at the rate of $\frac{4}{3} \text{ cm}^2/\text{sec}$

(d) Find the maximum value of y , if $y = 2x^3 - 9x^2 + 12x$

$$y = 2x^3 - 9x^2 + 12x$$

$$\frac{dy}{dx} = 6x^2 - 18x + 12$$

$$\frac{d^2y}{dx^2} = 12x - 18$$

$$\frac{dy}{dx} = 0$$

$$6x^2 - 18x + 12 = 0$$

$$6(x^2 - 3x + 2) = 0$$

$$x^2 - 3x + 2 = 0$$

$$(x - 2)(x - 1) = 0$$

$$x = 1, 2$$

$$\frac{d^2y}{dx^2} = 12x - 18$$

$$\text{At } x = 1, \frac{d^2y}{dx^2} = 12 - 18 = -6 < 0$$

\therefore at $x = 1$, y is maximum

And maximum value of $y = 2x^3 - 9x^2 + 12x$ is

$$y = 2x^3 - 9x^2 + 12x$$

$$y = 2(1)^3 - 9(1)^2 + 12(1)$$

$$y = 2 - 9 + 12$$

$$y = 5$$

(e) Evaluate

i. $\int \sqrt{1 + \sin 2x} \, dx$

ii. $\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}} dx$

i. $\int \sqrt{1 + \sin 2x} \, dx = \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x} \, dx$

$$\begin{aligned}
&= \int \sqrt{(\sin x + \cos x)^2} dx \\
&= \int \sin x + \cos x dx \\
&= -\cos x + \sin x \\
&= \sin x - \cos x + C
\end{aligned}$$

$$\text{ii. } \int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}} dx = \int u \frac{du}{2}$$

$$= \frac{1}{2} \int u du$$

$$= \frac{1}{2} \left[\frac{u^2}{2} \right]$$

$$= \frac{u^2}{4}$$

$$= \frac{(\sin^{-1} 2x)^2}{4} + C$$

$$\text{Let } u = \sin^{-1} 2x$$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-4x^2}} \times 2$$

$$\frac{du}{2} = \frac{1}{\sqrt{1-4x^2}} dx$$

(f) Find $\int (\log x)^2 dx$

$$\int (\log x)^2 x \, dx$$

$$= (\log x)^2 x - \int x \times 2(\log x) \frac{1}{x} dx$$

$$= (\log x)^2 x - 2 \int \log x \, dx$$

$$= (\log x)^2 x - 2 \left[\int (\log x \times 1) dx \right]$$

$$= (\log x)^2 x - 2 \left[\log x \times x - \int x \times \frac{1}{x} dx \right]$$

$$= x (\log x)^2 - 2 \log x \times x + 2 \int 1 dx$$

$$= x (\log x)^2 - 2x \log x + 2x + C$$

(g) Solve $\frac{dy}{dx} + \frac{x\sqrt{1+y^2}}{y\sqrt{1+x^2}} = 0$

$$\frac{dy}{dx} = - \frac{x\sqrt{1+y^2}}{y\sqrt{1+x^2}}$$

$$\frac{y dy}{\sqrt{1+y^2}} = -\frac{x}{\sqrt{1+x^2}} dx$$

Integrating on both sides

$$\int \frac{y dy}{\sqrt{1+y^2}} - \int \frac{x}{\sqrt{1+x^2}} dx$$

$$\int \frac{1}{\sqrt{v}} \frac{dv}{2} = - \int \frac{1}{\sqrt{u}} \frac{du}{2}$$

$$\frac{1}{2} \left[\frac{\sqrt{v}}{\frac{3}{2}} \right] = - \frac{1}{2} \left[\frac{\sqrt{u}}{\frac{3}{2}} \right]$$

$$\sqrt{v} = -\sqrt{u}$$

$$\sqrt{1+y^2} = \sqrt{1+x^2}$$

$$\sqrt{1+y^2} + \sqrt{1+x^2} = C$$

Put $1+x^2 = u$

$$\frac{du}{dx} = 2x$$

$$\frac{du}{2} = x dx$$

$$1+y^2 = v$$

$$\frac{dv}{dy} = 2y$$

$$\frac{dv}{2} = y dy$$

PART -C

(Maximum mark: 60)

Answer four full questions. Each question carries 15 marks.

III.

(a) Evaluate $\lim_{x \rightarrow 4} \frac{x-4}{x^2-16}$

$$\lim_{x \rightarrow 4} \frac{x-4}{x^2-16} = \frac{ma^{m-1}}{na^{n-1}}$$

$$m = 1$$

$$= \frac{1 \times 4^{1-1}}{2 \times 4^{2-1}}$$

$$n = 1$$

$$= \frac{1 \times 4^0}{2 \times 4^1}$$

$$a = 4$$

$$= \frac{1 \times 1}{2 \times 4}$$

$$= \frac{1}{8}$$

(b) Find $\frac{dy}{dx}$ if:

i. $y = \frac{e^{2x} \log x}{x^2}$

ii. $y = \frac{\tan^{-1} x}{(1+x^2)^2}$

i. $y = \frac{e^{2x} \log x}{x^2}$

$$\frac{dy}{dx} = \frac{x^2 \left[e^{2x} \frac{1}{x} + \log x e^{2x} \times 2 \right] - e^{2x} \log x \times 2x}{(x^2)^2}$$

$$\frac{dy}{dx} = \frac{[x^2 e^{2x} + 2x^2 \log x e^{2x} - 2x e^{2x} \log x]}{x^4}$$

ii. $y = \frac{\tan^{-1} x}{(1+x^2)^2}$

$$\frac{dy}{dx} = \frac{(1+x^2)^2 \times \frac{1}{(1+x^2)} - \tan^{-1} x \times 2(1+x^2) \times 2x}{(1+x^2)^4}$$

$$= \frac{(1+x^2)^2 [1 - 4x \tan^{-1} x]}{(1+x^2)^4}$$

$$= \frac{1 - 4x \tan^{-1} x}{(1+x^2)^3}$$

(c) find $\frac{dy}{dx}$, if x and y are connected by:

$$x = 3 \sin \theta - \sin^3 \theta$$

$$y = 3 \cos \theta - \cos^3 \theta$$

$$\frac{dx}{d\theta} = 3\cos\theta - 3\sin^3\theta \cos\theta$$

$$\frac{dy}{d\theta} = -3\sin\theta - 3\cos^2\theta \sin\theta - \sin\theta$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{-3\sin\theta + 3\sin\theta \cos^2\theta}{3\cos\theta - 3\sin^2\theta \cos\theta} \\ &= \frac{3\sin\theta(\cos^2\theta - 1)}{3\cos\theta(1 - \sin^2\theta)} \\ &= \tan\theta \times \frac{-\sin^2\theta}{\cos^2\theta} \\ &= -\tan^3\theta \end{aligned}$$

(d) If $2x^2 + 3xy + 5y^2 = 0$, find $\frac{dy}{dx}$

$$4x + 3x \frac{dy}{dx} + 3y + 10y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3x + 10y) = -4x - 3y$$

$$\therefore \frac{dy}{dx} = \frac{-4x - 3y}{3x + 10y}$$

IV.

(a) Using 1st principles, find the derivative of $\sin x$

$$y = \sin x$$

$$f(x) = \sin x$$

$$f(x + \Delta x) = \sin(x + \Delta x)$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \end{aligned}$$

$$\begin{aligned}
&= \lim_{\Delta x \rightarrow 0} \frac{2 \cos\left(\frac{x+\Delta x+x}{2}\right) \sin\left(\frac{x+\Delta x-x}{2}\right)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{2 \cos\left(x+\frac{\Delta x}{2}\right) \sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2} \times 2} \\
&= \cos\left(x + \frac{0}{2}\right) \times 1 \\
&= \cos x
\end{aligned}$$

(b) If $y = a \cos(\log x) + b \sin(\log x)$, prove that $x^2 y'' + xy' + y = 0$

$$\begin{aligned}
y' &= -a \sin(\log x) \times \frac{1}{x} + b \cos(\log x) \frac{1}{x} \\
xy' &= -a \sin(\log x) + b \cos(\log x) \\
xy'' + y' &= -a \cos(\log x) \frac{1}{x} + b \sin(\log x) \frac{1}{x} \quad \{\text{Differentiating on both sides}\} \\
x(xy'' + y') &= -(a \cos \log x + b \sin(\log x)) \\
x^2 y'' + xy' &= -y
\end{aligned}$$

ie, $x^2 y'' + xy' + y = 0$

(c) If $p = \frac{x}{x+R}$ find $\frac{d^2 p}{dx^2}$

$$\begin{aligned}
\frac{dp}{dx} &= \frac{(x+R) - x}{(x+R)^2} \\
&= \frac{x+R-x}{(x+R)^2}
\end{aligned}$$

$$\frac{dp}{dx} = \frac{R}{(x+R)^2}$$

$$\therefore \frac{d^2 p}{dx^2} = \frac{-2R}{(x+R)^3}$$

(d) If x and y are connected by $y \log x = x - y$. Prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$

$$y \log x = x - y$$

Differentiating on both sides we get

$$y + \frac{1}{x} + \log x \frac{dy}{dx} = 1 - \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} (\log x + 1) = 1 - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{\frac{x-y}{x}}{1+\log x}$$

$$\frac{dy}{dx} = \frac{\frac{y \log x}{x}}{1+\log x}$$

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

$$\left\{ \therefore \frac{y}{x} = \frac{1}{1+\log x} \right.$$

V.

- (a) Find the equation of the tangent and normal to the rectangular hyperbola $x = ct$, $y = \frac{c}{t}$ at $(ct, \frac{c}{t})$

$$x = ct \quad y = \frac{c}{t} \quad (\text{Question is wrong})$$

$$\frac{dx}{dt} = c \quad \frac{dy}{dt} = -\frac{c}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{-\frac{c}{t^2}}{c} = -\frac{1}{t^2}$$

$$\therefore \text{Slope } m = \frac{-1}{t^2}$$

- (b) Find the rate of change of volume of a cone with respect to radius when radius is equal to height

$$V = \frac{1}{3} \pi r^2 h$$

Given that $r = h$

$$\therefore V = \frac{1}{3} \pi r^2 \times r$$

$$V = \frac{1}{3} \pi r^3$$

$$\therefore \frac{dv}{dr} = \frac{1}{3} 3 \pi r^2$$

$$\frac{dv}{dr} = \pi r^2$$

- (c) An open tank is to be constructed with a square base and vertical side to hold a given quantity of water, show that the expense of lining it will be the least, if the depth is half the depth

Let base width = b

Height = h

$$\therefore \text{Surface area } S = b^2 + 4bh$$

$$\text{Given that } h = \frac{b}{2}$$

$$S = b^2 + 4b \times \frac{b}{2}$$

$$S = b^2 + 2b^2$$

$$S = 3b^2$$

$$\therefore \frac{ds}{db} = 6b$$

$$\frac{d^2s}{db^2} = 6b$$

$$\frac{d^2s}{db^2} > 0$$

$$\therefore S \text{ is minimum when } h = \frac{b}{2}$$

VI.

- (a) Prove that the function $x^3 + 6x^2 + 12x - 9$ is an increasing function for all values of x

$$y = x^3 + 6x^2 + 12x - 9$$

$$\frac{dy}{dx} = 3x^2 + 12x + 12$$

$$\frac{dy}{dx} = 3(x^2 + 4x + 4)$$

$$\frac{dy}{dx} = 3(x^2 + 2)^2 > 0 \text{ for all values of } x$$

- (b) A circular plate contracts when cooled. Find the rate of decrease in area if the radius is decreases at the rate of 0.1cm/min: when the radius is 4 cm.

$$\text{Area of the circle } A = \pi r^2$$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

$$= \pi \times 2 \times 4 \times 0.1$$

$$= 8\pi \times 0.1$$

$$\frac{dA}{dt} = .08\pi \text{ cm}^2/\text{min}$$

- (c) A particle is projected vertically upwards at its height 'h' feet at time 't' is given by $h = 60t - 16t^2$

$$h = 60t - 16t^2$$

$$\frac{dh}{dt} = 60 - 32t$$

$$\frac{d^2h}{dt^2} = -32 < 0$$

\therefore h is maximum

$$\frac{dh}{dt} = 0 \implies 60 - 32t$$

$$t = \frac{60}{32} = \frac{15}{8}$$

$$\therefore h = 60t - 16t^2$$

$$h = 60 \times \frac{15}{8} - 16 \times \left(\frac{15}{8}\right)^2$$

$$= \frac{15 \times 15}{2} - 16 \times \frac{225}{64}$$

$$= \frac{225}{2} - \frac{225}{4}$$

$$= \frac{450 - 225}{4}$$

$$= \frac{225}{4}$$

VII.

(a) Evaluate $\int \frac{2x^4}{1+x^{10}} dx$

$$\begin{aligned} \int \frac{2x^4}{1+x^{10}} dx &= \int \frac{2x^4}{1+(x^5)^2} dx \\ &= \frac{2}{5} \int \frac{1}{1+u^2} du \\ &= \frac{2}{5} \tan^{-1} u \\ &= \frac{2}{5} \tan^{-1} x^5 + C \end{aligned}$$

$$\begin{aligned} U &= x^5 \\ \frac{du}{dx} &= 5x^4 \\ \frac{du}{5} &= x^4 dx \end{aligned}$$

(b) Evaluate $\int_0^{\sqrt{3}} x\sqrt{1+x^2} dx$

$$\begin{aligned} \int_0^{\sqrt{3}} x\sqrt{1+x^2} dx &= \int_1^4 \sqrt{u} \frac{du}{2} \\ &= \frac{1}{2} \int_1^4 u^{\frac{1}{2}} du \\ &= \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_1^4 \\ &= \frac{1}{3} \left(4^{\frac{3}{2}} - 1^{\frac{3}{2}} \right) \\ &= \frac{1}{3} (8 - 1) \\ &= \frac{7}{3} \end{aligned}$$

$$\begin{aligned} u &= 1 + x^2 \\ \frac{du}{dx} &= 2x \\ \frac{du}{2} &= x dx \\ \text{When } x=0, u &= 1 \\ X=\sqrt{3}, u &= 1 + (\sqrt{3})^2 \\ &= 1 + 3 = 4 \end{aligned}$$

(c) Evaluate $\int (\tan x + \cot x)^2 dx$

$$\begin{aligned} \int (\tan x + \cot x)^2 dx &= \int (\tan^2 x + \cot^2 x + 2\tan x \cot x) dx \\ &= \int (\sec^2 x - 1 + \operatorname{cosec}^2 x - 1 + 2) dx \end{aligned}$$

$$\begin{aligned}
 &= \int (\sec^2 x + \operatorname{cosec}^2 x) dx \\
 &= \tan x - \cot x + C
 \end{aligned}$$

(d) Evaluate $\int \sqrt{1 + \sin 2x} dx$

$$\begin{aligned}
 \int \sqrt{1 + \sin 2x} dx &= \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x} dx \\
 &= \int \sqrt{(\sin x + \cos x)^2} dx \\
 &= \int (\sin x + \cos x) dx \\
 &= -\cos x + \sin x \\
 &= \sin x - \cos x + C
 \end{aligned}$$

(e) Evaluate $\int x e^{-x} dx$

$$\begin{aligned}
 \int x e^{-x} dx &= \frac{x e^{-x}}{-1} - \int \frac{e^{-x}}{-1} dx \\
 &= -x e^{-x} + \int e^{-x} dx \\
 &= -x e^{-x} - e^{-x} + C
 \end{aligned}$$

VIII.

(a) Evaluate $\int \frac{x^2}{(8+x^3)^4} dx$

$$\begin{aligned}
 &= \int \frac{1}{u^4} \frac{du}{3} \\
 &= \frac{1}{3} \int u^{-4} du \\
 &= \frac{1}{3} \frac{u^{-3}}{-3} \\
 &= \frac{u^{-3}}{-9}
 \end{aligned}$$

$$= \frac{(8+x^3)^{-3}}{-9} + C$$

$$\begin{aligned}
 u &= 8 + x^3 \\
 \frac{du}{dx} &= 3x^2 \\
 \frac{du}{3} &= x^2 dx
 \end{aligned}$$

(b) Evaluate $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx$

$$\begin{aligned}\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx &= \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x} dx \\ &= \int \frac{\cos^2 x}{\cos^2 x \sin^2 x} dx - \int \frac{\sin^2 x}{\cos^2 x \sin^2 x} dx \\ &= \int \frac{1}{\sin^2 x} dx - \int \frac{1}{\cos^2 x} dx \\ &= \int \operatorname{cosec}^2 x dx - \int \sec^2 x dx \\ &= -\cot x - \tan x + C\end{aligned}$$

(c) Evaluate $\int_0^\pi \frac{1}{1+\sin x} dx$

$$\begin{aligned}&= \int_0^\pi \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx \\ &= \int_0^\pi \frac{1-\sin x}{1-\sin^2 x} dx \\ &= \int_0^\pi \frac{1-\sin x}{\cos^2 x} dx \\ &= \int_0^\pi \frac{1}{\cos^2 x} dx - \int_0^\pi \frac{\sin x}{\cos^2 x} dx \\ &= \int_0^\pi \sec^2 x dx - \int_0^\pi \sec x \tan x dx \\ &= [\tan x - \sec x]_0^\pi \\ &= \tan \pi - \sec \pi - (\tan 0 - \sec 0)\end{aligned}$$

$$= 0 - 1 - (0 - 1)$$

$$= 0 + 1 + 1$$

$$= 2$$

(d) Evaluate $\int_0^2 x^2 \log x \, dx$

$$\begin{aligned}\int_0^2 x^2 \log x \, dx &= \int_0^2 \log x \, x^2 \, dx \\ &= \left[\log x \times \frac{x^3}{3} \right]_0^2 - \int_0^2 \frac{x^3}{3} \times \frac{1}{x} \, dx \\ &= \left[\frac{x^3}{3} \log x \right]_0^2 - \frac{1}{3} \int_0^2 x^2 \, dx \\ &= \left[\frac{x^3}{3} \log x \right]_0^2 - \frac{1}{3} \times \frac{x^3}{3} \\ &= \left(\frac{2^3}{3} \log 2 - \frac{0^3}{3} \log 0 \right) - \left(\frac{2^3}{9} - \frac{0^3}{9} \right) \\ &= \frac{8}{3} \log 2 - \frac{8}{9}\end{aligned}$$

IX.

(a) Find the area enclosed between the curve $y = x^2 - x - 2$ at the x - axis

$$y = x^2 - x - 2$$

$$\therefore x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x = 2, x = -1$$

$$\therefore \text{Area } A = \int_a^b y \, dx$$

$$= \int_{-1}^2 (x^2 - x - 2) \, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-1}^2$$

$$= \left(\frac{2^3}{3} - \frac{2^2}{2} - 2 \times 2 \right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2} - 2 \times (-1) \right)$$

$$= \left(\frac{8}{3} - \frac{4}{2} - 4 \right) - \left(\frac{-1}{3} - \frac{1}{2} + 2 \right)$$

$$\begin{aligned}
&= \left(\frac{8}{3} - 6\right) + \frac{5}{6} - 2 \\
&= \frac{8}{3} + \frac{5}{6} - 6 - 2 \\
&= \frac{16+5}{6} - 8 \\
&= \frac{21}{6} - 8 \\
&= \frac{7}{2} - 8 \\
&= -\frac{9}{2} \\
\therefore \text{Area} &= \frac{9}{2} \text{ unit}^2
\end{aligned}$$

(b) Find the volume generated when the point of parabola $y^2 = 4x$ between $x = 0$, and $x = 4$ revolves about the $x -$ axis.

$$\begin{aligned}
V &= \pi \int_a^b y^2 dx & Y^2 &= 4x \\
V &= \pi \int_0^4 4x dx & a &= 0, b= 4 \\
V &= \pi \left[\frac{4x^2}{2} \right]_0^4 \\
V &= \pi [2 \times 4^2 - 2 \times 0^2]_0^4 \\
V &= \pi [2 \times 16 - 0] \\
V &= 32\pi \text{ unit}^3
\end{aligned}$$

(c) Solve $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

$$\frac{dy}{dx} = e^x e^y + x^2 e^y$$

$$\frac{dy}{dx} = e^y (e^x + x^2)$$

$$\frac{dy}{e^y} = (e^x + x^2) dx$$

$$e^{-y} dy = (e^x + x^2) dx$$

Integrating on both sides, we get

$$-e^{-y} = e^x + \frac{x^3}{3} + c$$

X.

(a) Find the area enclosed between the curves $y = x^2$ and $2x + y - 3 = 0$

$$y = x^2 = f(x)$$

$$y = 3 - 2x = g(x)$$

$$\therefore x^2 = 3x - 2$$

$$x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0$$

$$x = -3, x = 1$$

$$\therefore A = \int_a^b (f(x) - g(x)) dx$$

$$= \int_{-3}^1 (x^2 - (3 - 2x)) dx$$

$$= \int_{-3}^1 x^2 + 2x - 3 dx$$

$$= \left[\frac{x^3}{3} + x^2 - 3x \right]_{-3}^1$$

$$= \left(\frac{1^3}{3} + 1^2 - 3 \times 1 \right) - \left(\frac{(-3)^3}{3} + (-3)^2 - 3 \times (-3) \right)$$

$$= \frac{1}{3} + 1 - 3 - \left(\frac{-27}{3} + 9 + 9 \right)$$

$$= \frac{1}{3} - 2 - (-9 + 9 + 9)$$

$$= \frac{1}{3} - 2 - 9$$

$$= \frac{1}{3} - 11$$

$$= -\frac{32}{3}$$

Area cannot be -ve.

$$\therefore \text{Area} = \frac{32}{3} \text{unit}^2$$

(b) Find the volume of the solid obtained by rotating one arch of the curve $y = \sin x$ about the x-axis.

$$y = \sin x, x = 0, \pi$$

$$\text{Volume } V = \int_a^b y^2 dx$$

$$V = \pi \int_0^\pi \sin^2 x dx$$

$$V = \pi \int_0^\pi \frac{1 - \cos 2x}{2} dx$$

$$= \frac{\pi}{2} \left[x - \frac{\sin 2x}{2} \right]_0^\pi$$

$$= \frac{\pi}{2} \left[\pi - \frac{\sin 2\pi}{2} - 0 + \frac{\sin 2 \times 0}{2} \right]$$

$$= \frac{\pi}{2} [\pi - 0 + 0 + 0]$$

$$= \frac{\pi^2}{2} \text{unit}^3$$

(c) Solve $x(1 + y^2)dx + y(1 + x^2)dy = 0$

$$x(1 + y^2)dx = -y(1 + x^2)dy$$

$$\frac{dy}{dx} = \frac{-x(1+y^2)}{y(1+x^2)}$$

$$\frac{y dy}{(1+y^2)} = \frac{-x}{(1+x^2)} dx$$

Integrating on both sides we get

$$\int \frac{y}{1+y^2} dy = - \int \frac{x}{1+x^2} dx$$

$$\int \frac{1}{u} \frac{du}{2} = - \int \frac{1}{v} \frac{dv}{2}$$

$$\frac{1}{2} \int \frac{1}{u} du = - \frac{1}{2} \int \frac{1}{v} dv$$

$$\frac{1}{2} \log u = - \frac{1}{2} \log v$$

$$\frac{1}{2} \log (1 + y^2) = - \frac{1}{2} \log (1 + x^2)$$

$$\frac{1}{2} \log (1 + y^2) = - \frac{1}{2} \log (1 + x^2)$$

$$\text{Put } 1 + y^2 = u$$

$$\frac{du}{dx} = 2y$$

$$\frac{du}{2} = y dy$$

$$V = 1 + x^2$$

$$\frac{dV}{dx} = 2x$$

$$\frac{dV}{2} = x dx$$

www.madinpoly.com